MATH 3235 Probability Theory

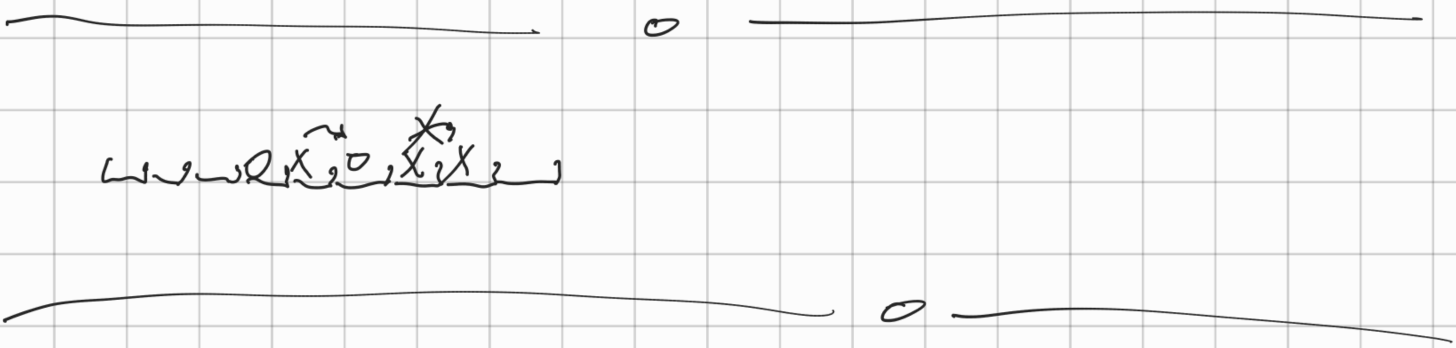
11/22/22

State space: discrete and finite

Musical chairs: $\{1 \ldots N\}$

Time is discrete

$X_1 \in \{1, \ldots N\}$



$X_0 \to X_1 \to X_2 \to X_3 \ldots$

$X_0 \quad X_1 \quad X_2 \quad \ldots \quad X_T$

r.v. with values in $\{1 \ldots N\}$.

Stochastic process

$X_1, \dots \quad X_T$ r.v.

$p(x_0, x_1, \dots, x_T)$ j.p.d.f of X_t

$p(x_0, x_1, \dots x_T)$ prob of the Trajectory $x_0, x_1 \dots x_T$.

Deterministic system gives prob 1 to 1 Trajectory.

Marginals T

$p(x,t) =$ p.d.f of X_t

$$\sum_{\substack{x_i \\ i \neq t}} p(x_0, x_1 \dots x_T) = p(x,t)$$

$$p(x_1, t_1 ; x_2, t_2) = \sum_{\substack{x_i \\ i = t_1, i = t_2}} p(x_0, x_1, \dots x_T)$$

$p(x_1, t_1; x_2, t_2; \dots; x_n, t_n)$

Compatibility relations:

$$\sum_{x_2} p(x_1, t_1; x_2, t_2) = p(x_1, t_1)$$

$p(x_1, t_1; x_2, t_2; \dots x_n, t_n)$

$\forall n \quad \forall t_1 \dots t_n \;\&\; \forall x_1 \dots x_n$.

uniquely define a stochastic process.

$p(x, t)$ does not depend on t

Stationary.

$f(x, y)$

$$\mathbb{E}(F(X_{t_1}, X_{t_2})) =$$

$$\sum_{x_0 \dots x_T} F(x_{t_1}, x_{t_2})\, p(x_0, x_1, \dots x_T)$$

for ~~any~~ $T \geq \max(t_1, t_2)$

Two point correlation

$$\mathbb{E}(X_{t_1} X_{t_2}) - \mathbb{E}(X_{t_1})\mathbb{E}(X_{t_2}) = C_{t_1 t_2}$$

$p(x_0, 0) = p_0(x_0)$ is the initial p.d.f.

$$p(x_0, 0; x_1; 1) = p(x_1, 1 \mid x_0, 0)\, p_0(x_0)$$

$$p(x_0, 0; x_1; 1; x_2, 2) =$$

$$p_0(x_0)\, p(x_1, 1 \mid x_0, 0)\, p(x_2, 2 \mid x_1, 1; x_0, 0)$$

$$p(x_0, 0; x_1; 1; x_2, 2; x_3, 3) =$$

$$p_0(x_0)\, p(x_1, 1 \mid x_0, 0)\, p(x_2, 2 \mid x_1, 1; x_0; 0)$$

$$p(x_3, 3 \mid x_2, 2, x_1, 1, x_0, 0).$$

Drastic Simplification:

$$p(x_T, T \mid x_{T-1}, T-1; \dots, x_0, 0) = p(x_T, t \mid x_{t-1}, t-1)$$

Markov Process.

$$T_t(x, y) = p(x, t+1 \mid y, t)$$

Transition Probability.

Stationary Markov Process

if $T_t(x, y)$ does not depend on t.

If my state space is $\{1 \dots N\}$ Then $T(x,y)$ can be seen an $N \times N$ matrix.

$p(x_0, x_1, \dots x_T)$ of a Trajectory from Time 1 to Time T

$$p(x_0, x_1, \dots x_T) =$$

$$p(x_0) T(x_1, x_0) T(x_2, x_1) T(x_3, x_2) \dots T(x_N, x_{N-1}) = p(x_0) \prod_{i=1}^{N} T_i(x_i, x_{i-1})$$

A stationary Markov process is defined by Two Things

1) initial probability $p_0(x)$

2) Transition probability

$T(x, y)$

prob of going from $y \to x$ in one Time step.

T Transition matrix

Transition Kernel

Random walk.

x_t position at time t

$T(x,y)$ probability of taking a step from $y \to x$

$T(x,y)$ depends only on $|x-y|$ with p.b.c.

$$\underset{1}{\circ} \quad \underset{2}{\circ} \quad \underset{3}{\circ} \quad \cdots \quad \underset{N-1}{\circ} \quad \underset{N}{\circ}$$

p_0

$\frac{p_1}{2}$ left $\frac{p_1}{2}$ right

$T =$